Perron-Frobenius Theorems

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This is a lecture note for Marxian Economic Theory, a course at Renmin University of China.

In this note, we will discuss the Perron-Frobenius Theorem, which is one of the most powerful tools on nonnegative/positive matrices and the workhorse in mathematical Marxian economics. I assume that the reader are familiar with basic linear algebra.

This Note is written in Pluto Notebook, a reactive notebook for Julia.

Nonnegative and Positive Matrices

Notations

In this note we use the following notations:

- A matrix is positive, $A > 0$, if and only if $a_{ij} > 0$ for all $i, j$.
- A matrix is semi-positive, $A \geq 0$, if and only if $a_{ij} \geq 0$ for all $i, j$ and $A \neq 0$.
- A matrix $A \geq 0$ if it is nonnegative.

As a vector can be taken as a special matrix ($1 \times k$ or $k \times 1$ matrix), the above notations apply to vectors.

About the dimensions: By default, a matrix is $n \times n$ and vector $n \times n$ or determined by the context. For example, the vector $v$ in $vA$ is by default a row vector, while $x$ is a column vector in $Ax$.

Example 1: Linear economy $(A, \ell)$

Nonnegative matrices arise in many fields. In Marxian Economic Theory, it is used in the model of linear economy $E(A, \ell)$.

Assume that there are $n$ goods in the economy, and each column of the matrix $A$ represents a production process.

$$a^j \oplus \ell_j \mapsto e_j$$
Specifically, $a_{ij}$ is the amount of good $i$ used in the production of 1-unit of good $j$. The row vector $\ell$ is the direct labor input, i.e., $\ell_j$ is the labor required in the production of 1-unit of good $j$.

The nonnegative matrix $A$ is called an input-output matrix. Usually, we assume the input-output matrix $A$ is productive and indecomposable, which will be discussed later.

**Example 2: Markov Chain**

Suppose that there are $n$ states, and the probability of jumping from state $j$ to state $i$ is given by $p_{ij} \geq 0$, then

$$\sum_{i=1}^{n} p_{ij} = 1, \text{ for all } j$$

The nonnegative matrix $P = (p_{ij})$ is called a stochastic matrix.

**Indecomposable Matrices**

A square matrix $A$ is decomposable if it can be reorganized by performing the same permutation on the rows and the columns into the form of

$$\begin{bmatrix}
A_{11} & A_{12} \\
0 & A_{22}
\end{bmatrix}$$

where $A_{11}$ and $A_{22}$ are square.

For example, if one column of the matrix $A$ is zero vector, say $a^1 = 0$, then $A$ is decomposable.

An equivalent definition: if the set $\{1, \ldots, n\}$ can be partitioned into two disjoint subsets $I$ and $J$ such that $a_{ij} = 0$ for all $i \in I$, $j \in J$, then $A$ is decomposable.

A square matrix $A$ is said to be indecomposable if it is not decomposable. In the context of Markov chains, it is also called irreducible.

Note that if a nonnegative matrix $A$ is indecomposable, then it must be semi-positive, $A \geq 0$.

If $A$ is indecomposable, the so is the transpose $A'$.

**Some Preliminary Properties**

- If $A > 0$, $x \geq 0$, then $Ax > 0$.
- If $A \geq 0$ is indecomposable, $x \geq 0$, then $Ax \geq 0$. For if $Ax = \sum_{i}^{n} x_i a^i = 0$, and $x_i > 0$, then $a^i = 0$ and $A$ is decomposable.
Eigenvalues and Eigenvectors of Nonnegative Matrices

The Perron-Frobenius Theorem

Theorem

Let $A \geq 0$ be indecomposable, then

1. $A$ has a positive eigenvalue $\lambda(A) > 0$ associated with a positive eigenvector $v > 0$

   $$Av = \lambda(A)v$$

2. If $u \geq 0$ is an eigenvector, then it has eigenvalue $\lambda(A)$, and $u > 0$ is a multiple of $v$.

3. If $\alpha$ is any eigenvalue of $A$, then $|\alpha| < \lambda(A)$.

4. If $A \geq B \geq 0$ then $\lambda(A) > \lambda(B)$. Moreover, if $A \geq B \geq 0$ and $\lambda(B) = \lambda(A)$, then $B = A$.

Statement (1) and (2) mean that $A$ processes a unique nonnegative eigenvector (up to a scalar), and its associated eigenvalue is positive. We call this unique eigenvalue $\lambda(A)$ the Frobenius root of the nonnegative matrix $A$.

Then (3) means that the Frobenius root is the largest eigenvalue, and (4) implies that the Frobenius root is an increasing and continuous function of the matrix.

If $A \geq 0$ without the assumption of indecomposability, then $\lambda(A) \geq 0$ and $v \geq 0$. Similarly, (3) and (4) hold with weak inequality.

Moreover, if $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ where $A_{11}$ is square. Consider $B = \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix}$ then

$$\lambda(A_{11}) = \lambda(B) < \lambda(A)$$

There are many different ways to prove this theorem. I present one of them using the fixed point theorem in the last section. Next let's first look at some examples.

- begin
  - import Pkg
  - Pkg.add("LinearAlgebra")
  - Pkg.add("Plots")
  - Pkg.add("LaTeXStrings")
  - using LinearAlgebra, Plots, LaTeXStrings
  - end
\[ A = 2 \times 2 \text{ Array}\{\text{Float64,}2\}: \]
\[
\begin{bmatrix}
0.5 & 0.2 \\
0.1 & 0.6
\end{bmatrix}
\]

\[ B = 2 \times 2 \text{ Array}\{\text{Float64,}2\}: \]
\[
\begin{bmatrix}
0.4 & 0.2 \\
0.1 & 0.6
\end{bmatrix}
\]

Let \( A \) be nonnegative, indecomposable, and \( \lambda(A) \) its Frobenius root, we have
• For an $x \geq 0$,

\[ Ax \leq sx \Rightarrow \lambda(A) \leq s \]
\[ Ax \geq sx \Rightarrow \lambda(A) \geq s \]

• For an $x > 0$,

\[ Ax \leq sx \Rightarrow \lambda(A) < s \]
\[ Ax \geq sx \Rightarrow \lambda(A) > s \]

The proofs of these four statements are identical. The readers are encouraged the prove them as an exercise. Below we present only the proof of the last one.

**Hint**

Let $v > 0$ be a eigenvector of the matrix $A'$ associated with $\lambda(A') = \lambda(A)$:

\[ A'v = \lambda(A)v \]

then $v'x > 0$ with $x \geq 0$. Since $Ax \geq sx$, then

\[ v'Ax > sv'x \Rightarrow \lambda(A)v'x > sv'x \Rightarrow \lambda(A) > s. \]

If $A$ is nonnegative, without the assumption of indecomposability, then we have

• For an $x > 0$,

\[ Ax \leq sx \Rightarrow \lambda(A) \leq s \]
\[ Ax \geq sx \Rightarrow \lambda(A) \geq s \]

• For an $x \geq 0$,

\[ Ax < sx \Rightarrow \lambda(A) < s \]
\[ Ax > sx \Rightarrow \lambda(A) > s \]

**Max-min and Min-max Characterization**
Let $A$ be nonnegative, indecomposable and $\lambda(A)$ its Frobenius root. It is immediately from the above observations that

- if $w > 0$, $Aw \leq \lambda(A)w$, then $Aw = \lambda(A)w$
- if $w > 0$, $Aw \geq \lambda(A)w$, then $Aw = \lambda(A)w$

In other words, it is impossible to have

$$Aw \leq \lambda(A)w$$

or

$$Aw \geq \lambda(A)w$$

for any $w > 0$. Then, $\lambda(A)$ cannot be the minimum nor the maximum of the ratio $\frac{(Aw)_i}{w_i}$.

Therefore, we have the following theorem

**Theorem**

Let $A$ be nonnegative, indecomposable and $\lambda(A)$ its Frobenius root. Let $w > 0$, then either

$$\min_i \frac{(Aw)_i}{w_i} < \lambda(A) < \max_i \frac{(Aw)_i}{w_i},$$

or

$$\min_i \frac{(Aw)_i}{w_i} = \lambda(A) = \max_i \frac{(Aw)_i}{w_i}.$$

For example, consider the $w > 0$ above, then

- the minimum of $\frac{(Aw)_i}{w_i}$ is 0.64;
- the maximum of $\frac{(Aw)_i}{w_i}$ is 1.0;
- while $\lambda(A) = 0.7$ is in between.
Productive Matrix

A nonnegative square matrix is said to be productive if there exists \( x \geq 0 \) such that

\[
x - Ax \geq 0
\]

**Exercise.**

Let \( A \geq 0 \) be indecomposable. Show that \( A \) is productive if and only if \( \lambda(A) < 1 \).

**Hint**

If \( A \) is productive, then there exists \( x \geq 0 \) such that \( Ax \leq x \). Therefore, \( \lambda(A) < 1 \).

If \( \lambda(A) < 1 \), then we have \( v > 0 \) and \( Av = \lambda v < v \), i.e., \( v - Av > 0 \), therefore, \( A \) is productive.

**Theorem**

Let \( A \geq 0 \) be indecomposable, then \( s > \lambda(A) \) if and only if \( (sI - A)^{-1} > 0 \).

**Proof**

By the Perron-Frobenius Theorem, \( \lambda(A) \) is the largest eigenvalue of \( A \), i.e., the largest root of the equation

\[
\det(\lambda I - A) = 0
\]

Then \( s > \lambda(A) \) is not a root, i.e., \( \det(sI - A) \neq 0 \). Therefore, \( (I - A)^{-1} \) exists.

Next, we show that \( (sI - A)^{-1} > 0 \). It is sufficient to show that \( z = (sI - A)^{-1}y > 0 \) for any \( y \geq 0 \). Suppose that \( z \) has some negative components,

\[
z = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}
\]

where \( x_1 > 0 \) and \( x_2 \geq 0 \), then

\[
\begin{bmatrix}
sI - A_{11} & -A_{12} \\
-A_{21} & sI - A_{22}
\end{bmatrix}
\begin{bmatrix}
-x_1 \\
x_2
\end{bmatrix}
= y
\]
implies \(-(sI - A_{11})x_1 - A_{12}x_2 \geq 0\). Then \((sI - A_{11})x_1 \geq 0\) and therefore \(\lambda(A_{11}) \geq s\), a contradiction to the fact that \(\lambda(A_{11}) \leq \lambda(A) < s\). Therefore, we have \(z \geq 0\).

Suppose that \(z\) has zero components, then
\[
\begin{bmatrix}
    sI - A_{11} & -A_{12} \\
    -A_{21} & sI - A_{22}
\end{bmatrix}
\begin{bmatrix}
    0 \\
    x_2
\end{bmatrix}
= y
\]
where \(x_2 > 0\). Then
\[-A_{12}x_2 \geq 0 \Rightarrow A_{12}x_2 = 0\]
and then \(A_{12} = 0\) since \(x_2 > 0\), violating the indecomposability of \(A\). Therefore, \(z > 0\).

Applying the above theorem to the case with \(\lambda(A) < 1\) when \(A\) is productive, we have

**Take-Home Message**

If a nonnegative matrix \(A\) is indecomposable and productive, then \((I - A)^{-1} > 0\).

Moreover,
\[
(I - A)^{-1} = I + A + A^2 + \cdots = \sum_{k=0}^{\infty} A^k
\]
holds when \(\lambda(A) < 1\), as a generalization of
\[
\frac{1}{1 - q} = 1 + q^2 + \cdots = \sum_{k=0}^{\infty} q^k, \quad \text{for } |q| < 1
\]

**Appendix: The proof of the Perron-Frobenius Theorem**

**Theorem**

Let \(A \geq 0\) be indecomposable, then

1. \(A\) has a positive eigenvalue \(\lambda(A) > 0\) associated with a positive eigenvector \(v > 0\)
\[ Av = \lambda(A)v \]

(2) If \( u \geq 0 \) is an eigenvector, then it has eigenvalue \( \lambda(A) \), and \( u > 0 \) is a multiple of \( v \).

(3) If \( \alpha \) is any eigenvalue of \( A \), then \( |\alpha| < \lambda(A) \).

(4) If \( A \geq B \geq 0 \) then \( \lambda(A) > \lambda(B) \). Moreover, if \( A \geq B \geq 0 \) and \( \lambda(B) = \lambda(A) \), then \( B = A \).

We first establish (1) using the following fixed point theorem.

Let \( \Delta = \{ x \geq 0 \mid x \in \mathbb{R}^n \text{ and } \|x\| = 1 \} \), and \( f : \Delta \to \Delta \) be a continous function. The Brouwer fixed point theorem ensures that \( f \) has a fixed point, i.e., there exists \( x_0 \in \Delta \) such that \( f(x_0) = x_0 \).

For a nonnegative and indecomposable matrix \( A \), we first show that there exist a positive eigenvalue \( \lambda > 0 \) such that

\[ Av = \lambda v \]

Since \( A \) is indecomposable, for any \( x \geq 0 \), we have \( Ax \geq 0 \), then \( \|Ax\| > 0 \). Define \( T : \Delta \to \Delta \) by

\[ T(x) = \frac{Ax}{\|Ax\|}, \forall x \in \Delta \]

then \( T \) is continous and there exists a fixed point \( v \in \Delta \) such that

\[ T(v) = \frac{Av}{\|Av\|} = v \Rightarrow Av = \lambda v \]

where \( \lambda = \|Av\| > 0 \).
Next, we show that \( v \) must be positive by the proof of contradiction. Without loss of generality, suppose on the contrary that \( v \) has some zero components,

\[
v = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}
\]

where \( v_1 > 0 \). Then \( Av = \lambda v \) yields

\[
\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda v_1 \\ 0 \end{bmatrix}
\]

Thus \( A_{21}v_1 = 0 \), so \( A_{21} = 0 \), violating the indecomposability of \( A \).
For (2), let \( w > 0 \) be an eigenvector of \( A' \) associated with \( \lambda(A') = \lambda(A) \). Suppose that \( Au = \lambda_u u \), then

\[
\lambda_u w' u = w' Au = \lambda(A) w' u
\]

so \( \lambda_u = \lambda(A) \) since \( w' u > 0 \). Then \( u > 0 \) by the same argument (showing \( v > 0 \)) above.

Suppose that \( u \) is not a multiple of \( v \), then there exist \( k \in \mathbb{R} \) such that \( u + kv \geq 0 \) has zero component. Note that \( u + kv \) is an eigenvector associated with \( \lambda(A) \), so \( u + kv > 0 \) contradicted.